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On a result of Cartlidge

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Abstract

We give a simple proof of Cartlidge's result on the l^p operator norms of weighted mean matrices.

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1. Introduction

Suppose throughout that $p \neq 0$, $\frac{1}{p} + \frac{1}{q} = 1$. Let l^p be the Banach space of all complex sequences $\mathbf{a} := (a_n)_{n \geqslant 1}$ with norm

$$\|\mathbf{a}\|_p := \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} < \infty.$$

We say a matrix $A = (a_{n,k})$ is a weighted mean matrix if its entries satisfy:

$$a_{n,k} = \begin{cases} \lambda_k / \Lambda_n, & 1 \leqslant k \leqslant n, \\ 0, & k > n, \end{cases} \tag{1.1}$$

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where $\lambda_1 > 0$, $\lambda_k \geqslant 0$ and $\Lambda_n = \sum_{k=1}^{n} \lambda_k$. We define the l^p operator norm of A to be

$$\|A\|_{p,p} := \sup_{\|\mathbf{a}\|_p \leqslant 1} \left(\sum_{n=1}^{\infty} |A_n|^p \right)^{1/p},$$

where

$$A_n = \sum_{k=1}^{\infty} a_{n,k} a_k.$$

In an unpublished dissertation [4], Cartlidge studied weighted mean matrices as operators on l^p and obtained the following result (see also [1, p. 416, Theorem C]).

Theorem 1.1. *Let $1 < p < \infty$ be fixed. Let A be a weighted mean matrix given by* (1.1). *If*

$$L = \sup_n \left(\frac{\Lambda_{n+1}}{\lambda_{n+1}} - \frac{\Lambda_n}{\lambda_n} \right) < p, \tag{1.2}$$

then $\|A\|_{p,p} \leqslant p/(p-L)$.

From now on we assume $a_n > 0$ and any infinite sum converges. The most important special case of Theorem 1.1, obtained by setting $\lambda_n = 1$, is the celebrated Hardy's inequality [7, Theorem 326], which asserts that for $p > 1$,

$$\sum_{n=1}^{\infty} A_n^p \leqslant \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p. \tag{1.3}$$

We note that Hardy's inequality follows from the following result of Elliott [5]:

$$\sum_{n=1}^{\infty} A_n^p \leqslant \left(\frac{p}{p-1} \right) \sum_{n=1}^{\infty} a_n A_n^{p-1}. \tag{1.4}$$

In fact, by Hölder's inequality, one has

$$\sum_{n=1}^{\infty} a_n A_n^{p-1} \leqslant \left(\sum_{n=1}^{\infty} a_n^p \right)^{1/p} \left(\sum_{n=1}^{\infty} A_n^p \right)^{1-1/p}.$$

Motivated by Elliott's result, as well as the works of Broadbent [3], Grandjot [6] and Redheffer [8], we will present in this paper a proof of the following result:

Theorem 1.2. *Let $1 < p < \infty$ be fixed. Let A be a weighted mean matrix given by* (1.1). *If* (1.2) *is satisfied, then*

$$\sum_{n=1}^{\infty} A_n^p \leqslant \left(\frac{p}{p-L} \right) \sum_{n=1}^{\infty} a_n A_n^{p-1}. \tag{1.5}$$

In particular, $\|A\|_{p,p} \leqslant p/(p-L)$.

We note here Theorem 1.2 implies Theorem 1.1 just as (1.4) implies (1.3). We also refer the reader to the Borwein's paper [2], where he proved a far more general result than Theorem 1.1.

2. Proof of Theorem 1.2

It suffices to prove the theorem for any finite summation from $n = 1$ to N with $N \geqslant 1$. If (1.2) is satisfied then $\lambda_n > 0$ for any n. We start with the inequality $x^p - px + p - 1 \geqslant 0$. By setting $x = A_{n-1}/A_n$ for $n \geqslant 2$, we obtain

$$A_{n-1}^p + (p-1)A_n^p \geqslant pA_{n-1}A_n^{p-1}. \tag{2.1}$$

Note that

$$A_{n-1} = \frac{\Lambda_n}{\Lambda_{n-1}}A_n - \frac{\lambda_n}{\Lambda_{n-1}}a_n.$$

Substitute this for the A_{n-1} on the right-hand side of (2.1), we obtain after some simplifications that

$$(\Lambda_n/\lambda_n + p - 1)A_n^p - (\Lambda_n/\lambda_n - 1)A_{n-1}^p \leqslant pa_nA_n^{p-1}. \tag{2.2}$$

By defining $A_0 = 0$ the above inequality also holds for $n = 1$.

Summing (2.2) from $n = 1$ to N gives

$$(\Lambda_N/\lambda_N + p - 1)A_N^p + \sum_{n=1}^{N-1}(\Lambda_n/\lambda_n - \Lambda_{n+1}/\lambda_{n+1} + p)A_n^p \leqslant p\sum_{n=1}^{N} a_nA_n^{p-1}. \tag{2.3}$$

By condition (1.2), $\Lambda_n/\lambda_n - \Lambda_{n+1}/\lambda_{n+1} + p \geqslant p - L$. Inequality (1.5) hence follows from (2.3) and this completes the proof.

3. Further discussions

We remark here for $p > 1$ being an integer, writing $x^p - px + p - 1 = (x-1)^2(x^{p-2} + 2x^{p-3} + \cdots + p - 1)$, substituting A_{n-1}/A_n for x and proceeding as in the previous section, one can deduce Theorem 1.2 from identities. This generalizes an approach of Elliott in [5] (see also [6]). For example, consider the case $\lambda_n = 1$ and $p = 2$, we have

$$(n-1)A_{n-1}^2 + 2a_nA_n - (n+1)A_n^2 = (n-1)(A_{n-1} - A_n)^2.$$

Summing from $n = 1$ to N gives

$$\sum_{n=1}^{N}\bigl(2a_nA_n - A_n^2\bigr) = \sum_{n=1}^{N-1}(n-1)(A_{n-1} - A_n)^2 + NA_N^2 \geqslant 0,$$

which is essentially Elliott's approach.

Now let $1 < p < \infty$ be fixed. Let A be a weighted mean matrix given by (1.1) and suppose (1.2) is satisfied. For a fixed $N \geqslant 1$, we define

$$f(t) = \left(\frac{p}{p-L}\right)^t \sum_{n=1}^{N} a_n^t A_n^{p-t}.$$

Proposition 3.1. *The function $f(t)$ defined above is convex. For $t \geqslant 1$, $f(t)$ is increasing and $f(t) \geqslant f(0)$.*

Proof. One checks easily that $f''(t) \geqslant 0$ and $f(t)$ is hence convex. Using the inequality $\ln x \geqslant 1 - 1/x$, we have

$$\left(\frac{p-L}{p}\right) f'(1) = \sum_{n=1}^{N} a_n A_n^{p-1} \ln \frac{pa_n}{(p-L)A_n} \geqslant \sum_{n=1}^{N} a_n A_n^{p-1} \left(1 - \frac{(p-L)A_n}{pa_n}\right) \geqslant 0,$$

where the last inequality follows from Theorem 1.2. Since $f(t)$ is convex, this shows $f(t)$ is increasing for $t \geqslant 1$. Also by Theorem 1.2, we have $f(t) \geqslant f(1) \geqslant f(0)$, which completes the proof. □

We note here $f'(t) \geqslant 0$ does not in general hold for $0 \leqslant t < 1$ as the counterexample $t = 0$, $N = 2$, $a_1 = 1$, $a_2 \to 0$ shows. However, in the limiting case $p \to \infty$, we do have a better result. In this case we define

$$g(t) = e^{tL} \sum_{n=1}^{N} a_n^t G_n^{1-t},$$

where $G_n = \prod_{k=1}^{n} a_k^{\lambda_k/\Lambda_n}$.

Theorem 3.1. *The function $g(t)$ defined above is convex and $g(t)$ is increasing for $t \geqslant 0$.*

Proof. It is easy to see that $g(t)$ is convex. Now using the inequality $e^x - 1 - x \geqslant 0$ and substituting $\ln(G_{n-1}/G_n)$, $n \geqslant 2$, for x, we obtain

$$G_n \ln \frac{G_n}{G_{n-1}} - G_n + G_{n-1} = \frac{\lambda_n}{\Lambda_{n-1}} G_n \ln \frac{a_n}{G_n} - G_n + G_{n-1} \geqslant 0.$$

We rewrite the inequality above as

$$G_n \ln \frac{a_n}{G_n} - \left(\frac{\Lambda_n}{\lambda_n} - 1\right) G_n + \left(\frac{\Lambda_n}{\lambda_n} - 1\right) G_{n-1} \geqslant 0.$$

Summing the above from $n = 1$ to N gives (where we set $G_0 = 0$)

$$\sum_{n=1}^{N} G_n \ln \frac{a_n}{G_n} + \sum_{n=1}^{N-1} \left(\frac{\Lambda_{n+1}}{\lambda_{n+1}} - \frac{\Lambda_n}{\lambda_n}\right) G_n - \left(\frac{\Lambda_N}{\lambda_N} - 1\right) G_N \geqslant 0.$$

By (1.2), we then have

$$g'(0) = \sum_{n=1}^{N} G_n \ln \frac{e^L a_n}{G_n} \geqslant 0.$$

Since $g(t)$ is convex, this shows $g(t)$ is increasing for $t \geqslant 0$ and the proof is completed. □

We note here Theorem 3.1 implies $g(1) \geqslant g(0)$, the well-known Carleman's inequality. In fact, the inequality $g'(0) \geqslant 0$ along implies $g(1) \geqslant g(0)$ since $g(t)$ is convex. One can also see this directly by using the inequality $x - 1 \geqslant \ln x$ so that

$$\sum_{n=1}^{N} G_n \left(\frac{e^L a_n}{G_n} - 1\right) \geqslant \sum_{n=1}^{N} G_n \ln \frac{e^L a_n}{G_n} \geqslant 0.$$

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